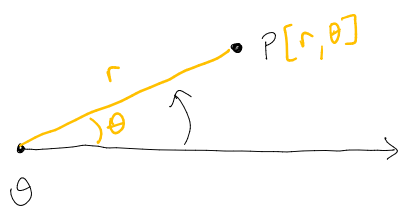
Section 8.5

Thursday, May 28, 2020 6:57 PM

Polar coordinates and polar curves

In polar coordinate system, we choose a reference point, the origin, and a polar axis which is a ray starting from the origin and extending indefinitely to "right". Given any point P in the plane, we define its polar coordinates of P to $[r,\theta]$ where



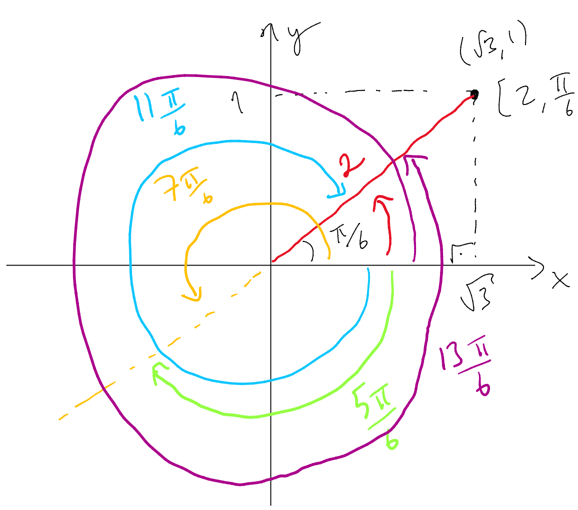
- r is the distance of P to O, and
- θ is the angle between the polar axis and the line segment OP measured from the polar axis in the counterclockwise direction.

So the point with polar coordinates $[r,\theta]$ in the plane is found as follows:

- Find the ray starting from the origin whose angle with the polar axis measured from the polar axis in the counterclockwise direction is θ
- Move r units on this ray in the direction of the ray

It is convenient to allow r or θ to negative, in which case we interpret the minus sign as "moving in the opposite direction".

Remark: When one wants to use both polar and cartesian coordinate systems, then the polar axis is usually taken to be the x-axis.

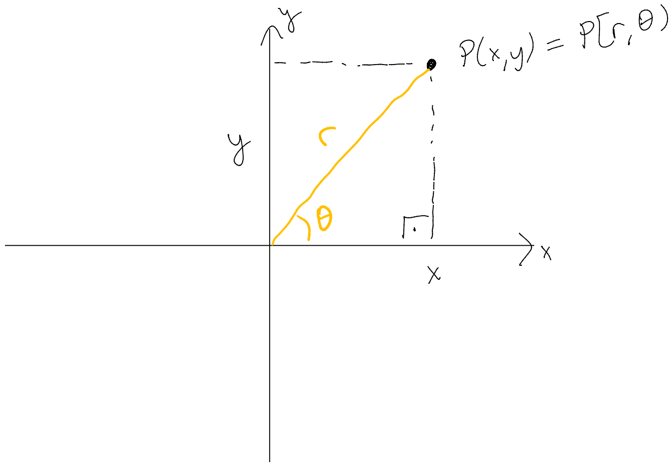


$$[2,\tfrac{\pi}{6}] = [-2,\tfrac{7\pi}{6}] = [2,\tfrac{-11\pi}{6}] = [-2,\tfrac{-5\pi}{6}] = [2,\tfrac{13\pi}{6}]$$

As it can be seen, the polar coordinates of a point are not unique. For example,

$[r,\theta] = [r,\theta+2\pi k]$ for any integer k.

How do we change our coordinate system between cartesian and polar coordinates?

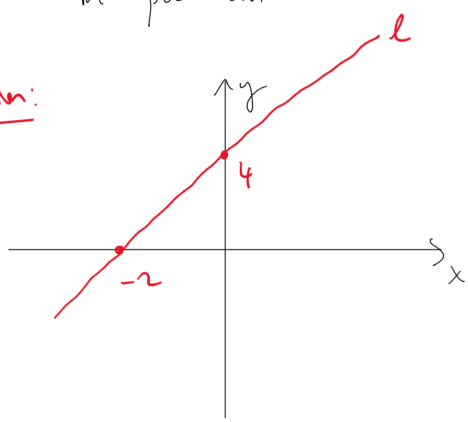


$$\boxed{\begin{array}{l} r^2 = x^2 + y^2 \\ \tan\theta = \dfrac{y}{x} \end{array}}$$

$$\boxed{\begin{array}{l} y = r\sin\theta \\ x = r\cos\theta \end{array}}$$

Example: Find the equation of the line given by $y - 2x - 4 = 0$ (in cartesian coordinates) in polar coordinates.

Solution:



The equation of ℓ in polar coordinates is

$$y - 2x - 4 = 0$$

$$r\sin\theta - 2r\cos\theta - 4 = 0$$

$$r = \frac{4}{\sin\theta - 2\cos\theta} = f(\theta)$$

Example: Find the equation of the circle centered at $(1,0)$ with radius 1 in polar coordinates.

Solution I: The equation of this circle in cartesian coordinates is $(x-1)^2 + y^2 = 1$

So its polar equation is

$$(x-1)^2 + y^2 = 1$$

$$x^2 - 2x \cancel{+1} + y^2 = \cancel{1}$$

$$x^2 + y^2 = 2x$$

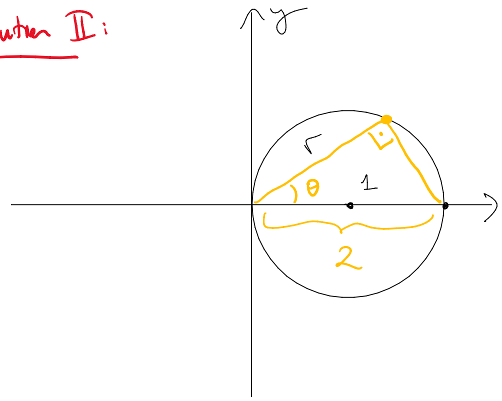
$$r^2 = 2r\cos\theta$$

$$r = 2\cos\theta$$

if $r = 0$, then the corresponding point is the origin which is still obtained by the second equation for $\theta = \pi$

So we are not "losing" a point by this cancellation.

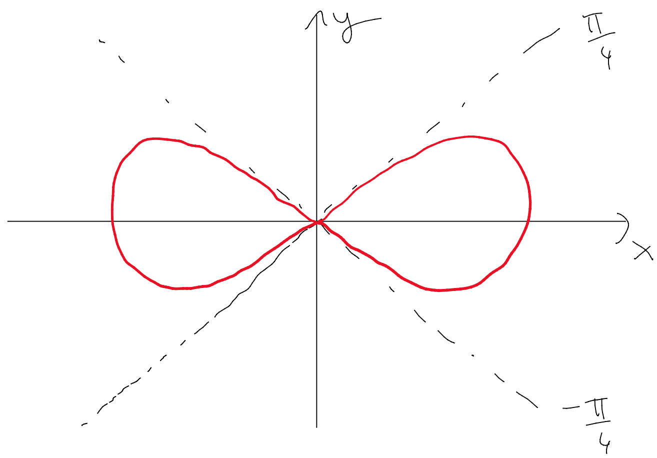
Solution II:



$$r = 2\cos\theta$$

Example: Find the equation of Bernoulli's lemniscate given in polar coordinates by $r^2 = \cos(2\theta)$ in cartesian coordinates.

Solution:



The equation of this in cartesian coordinates is

$$r^2 = \cos(2\theta)$$

$$r^2 = \cos^2\theta - \sin^2\theta$$

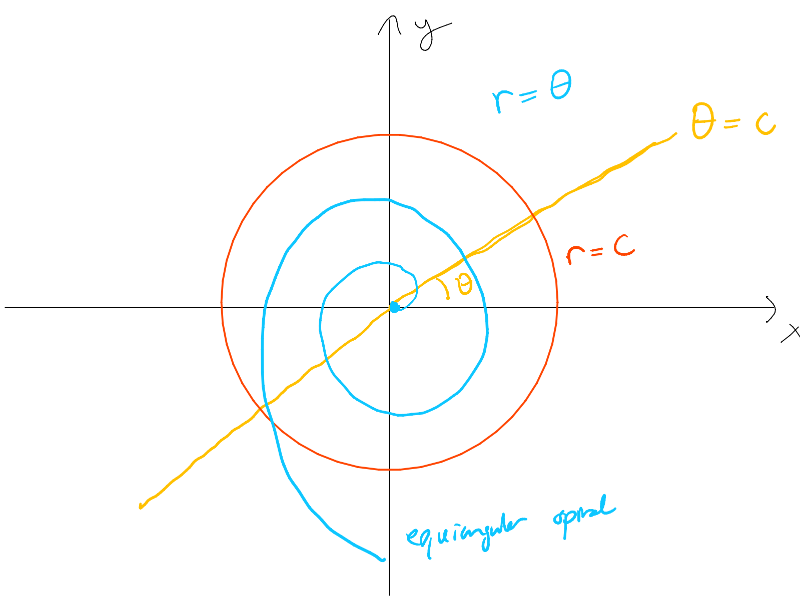
$$r^4 = r^2\cos^2\theta - r^2\sin^2\theta$$

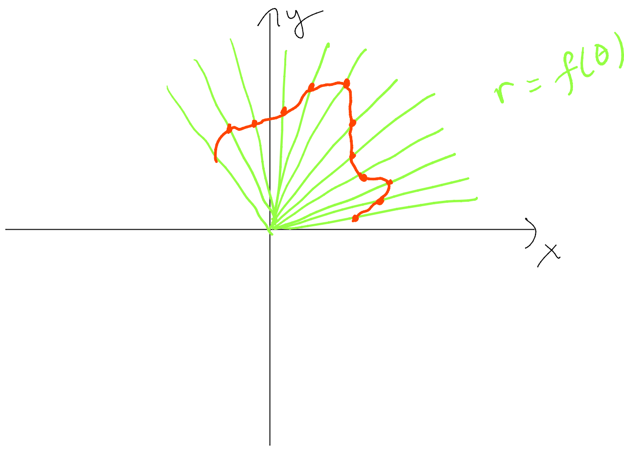
$$(r^2)^2 = (r\cos\theta)^2 - (r\sin\theta)^2$$

$$(x^2+y^2)^2 = x^2 - y^2$$

Some polar curves defined by some basic equations

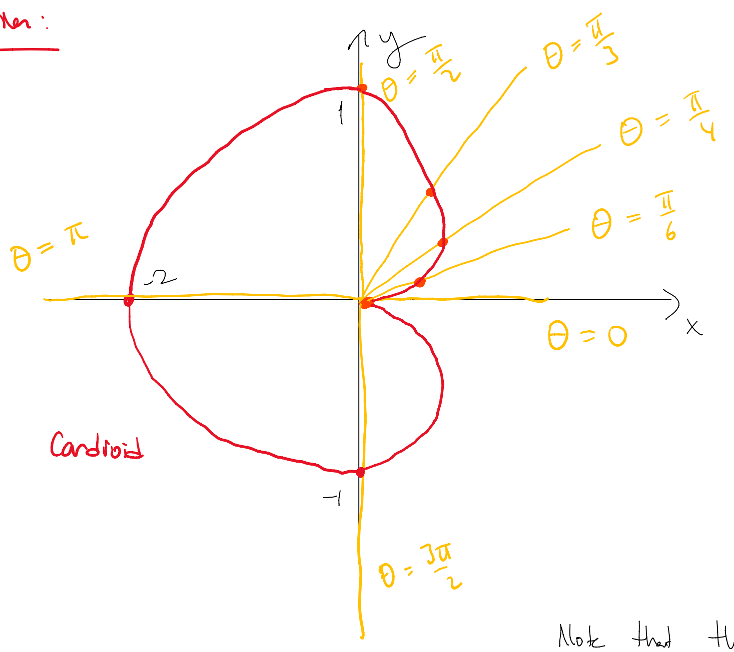
c constant





Example: Sketch the curve given by $r = 1 - \cos\theta$ in polar coordinates.

Solution:

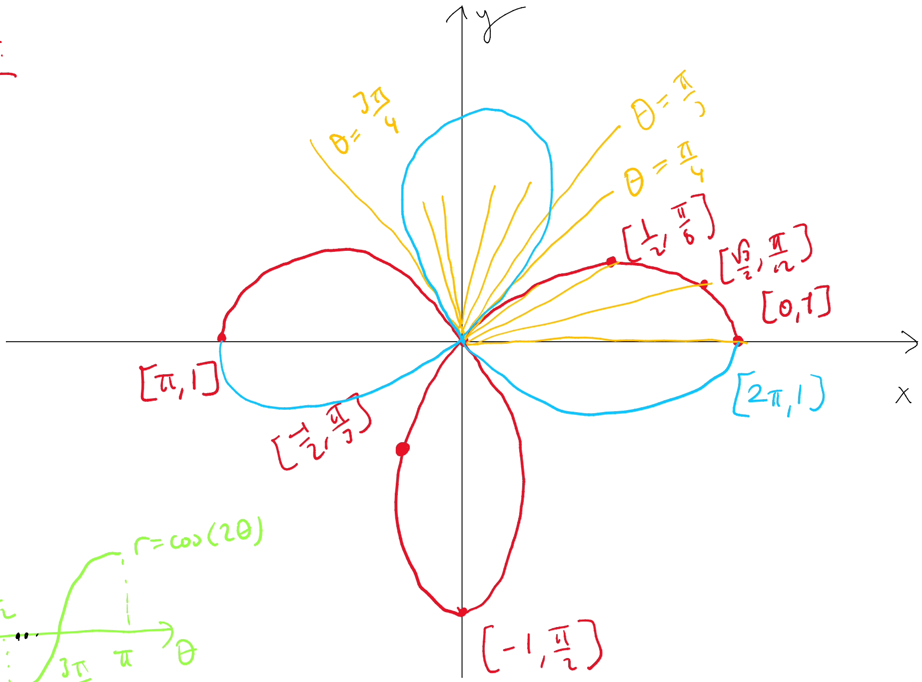


θ	r
0	0
$\frac{\pi}{6}$	$1 - \frac{\sqrt{3}}{2}$
$\frac{\pi}{4}$	$1 - \frac{\sqrt{2}}{2}$
$\frac{\pi}{3}$	$\frac{1}{2}$
$\frac{\pi}{2}$	1
$\vdots$	$\vdots$
π	-2

Note that this curve is going to be symmetric with respect to the x-axis because if $[r, \theta]$ satisfies $r = 1 - \cos\theta$, then so does $[r, -\theta]$. Also, it suffices to plot the points for $0 \le \theta < 2\pi$ since $\cos(\theta) = \cos(\theta + 2\pi k)$ for all integers k.

Example: Sketch the curve given by $r = \cos(2\theta)$ in polar coordinates

Solution:



θ	r
0	1
$\frac{\pi}{12}$	$\frac{\sqrt{3}}{2}$
$\frac{\pi}{6}$	$\frac{1}{2}$
$\frac{\pi}{4}$	0
$\frac{\pi}{3}$	$-\frac{1}{2}$
$\vdots$	$\vdots$
$\frac{\pi}{2}$	-1
$\vdots$	$\vdots$
π	1

Observe that if $[r, \theta]$ satisfies $r = \cos(2\theta)$, then so does $[r, \theta + \pi]$. Thus the curve defined by this equation is symmetric with respect to the origin.